Construction of Instantons

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Contents

Yang-Mills Theory

Yang-Mills theory is a generalization of Maxwell's theory on electromagnetism, used to describe the weak force and the strong force in subatomic particles. This theory was introduced by physicists Yang C.N. and Robert L.Mills.

Surprisingly, Simons and Yang discovered the correspondences between Yang-Mills theory and fiber bundle theory: gauge potential to connection on a principal bundle, gauge field to curvature, electromagnetism to connections on $U(1)$-bundle, Dirac's monopole quantization to classfication of $U(1)$-bundle and so on. Physicists and mathematicians developed their theory independently but finally coincided.

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Given a vector bundle E(complex or real) over M, a connection on E is a map:

$$\nabla_A : \Omega^0_M(E) \to \Omega^1_M(E)$$

which satisfies the Leibnitz rule:

$$\nabla_A(fs) = df \otimes s + f\nabla_A s$$

Here Ω^p_M denotes sections of $\bigwedge^p T^*M \otimes E$, i.e. p-forms with value in E. A connection naturally induces a differential operator called covariant derivative $d_A : \Omega^p_M(E) \to \Omega^{p+1}_M(E)$, and curvature of ∇_A is defined to be $F_A = d_A^2$ on $\Omega^0_M(E)$.

Yang-Mills Theory

Given E a Hermitian bundle over a Riemannian manifold M, we can define the Yang-Mills functional of a given connection ∇_A as follows:

$$\|F_A\|^2 = \int_M |F_A|^2 d\mu$$

And we can obtain its Euler-Lagrange equation:

$$d_A^* F_A = 0$$

which is known as **Yang-Mills equation**. When structure group is the abelian group $U(1)$, this equation is just classical Maxwell equation.

Instantons

Now we focus on $SU(2)$-bundle E over compact 4-manifolds. Then for any $SU(2)$-connections A, first Chern class must vanish and E is topologically determined by second Chern class given by:

$$c_2(E) = \frac{1}{8\pi^2}[Tr(F_A \wedge F_A)]$$

Equivalently, we can consider the second Chern number; but here we refer it as Pontrjagin index, following physicists' terminology:

$$k = -c_2 = -\frac{1}{8\pi^2}\int_M Tr(F_A \wedge F_A)$$

Instantons

By Hodge $*$ operator, we divide the integral into two parts:

$$k = \frac{1}{8\pi^2}(\|F_A^+\|^2 - \|F_A^-\|^2)$$

which gives a lower bound of Yang-Mills functional:

$$\|F_A\|^2 = \|F_A^+\|^2 + \|F_A^-\|^2 \geqslant 8\pi^2|k|$$

This minimum is obtained iff $F_A^- = 0$ or $F_A^+ = 0$, depending on whether k is positive or not. We then call them self-dual connections or anti-self-dual connections respectively. By physical consideration, we also call them **instantons**. Note that instanton satisfies Yang-Mills equation trivially.

Instanton Moduli Space

The set of all instantons moduli the action of gauge transformations becomes a smooth manifold, possibly with some singularities. A typical example is the moduli space over S^4.

Theorem (Atiyah-Hitchin-Singer,1977)

The space of moduli of self-dual $SU(2)$-Yang-Mills fields over S^4, with Pontrjagin index $k \geqslant 1$, is a manifold of dimension $8k - 3$.

Later, these parameters are constructed explicitly via just linear algebra, named as "ADHM Construction", given by Atiyah,Hitchin,Drinfeld and Manin. Their construction is motivated from algebraic geometry.

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The most surprising fact is that the study of instanton moduli space can be **applied** to differential topology research.

Wild at Heart

Dimension 4 is mysterious and wild. You may consider the following facts: Let M^n be a closed topological manifold, then

1. If $n \leqslant 3$, there is exactly one smooth structure on M;
2. If $n \geqslant 5$, there are at most finitely many smooth structures on M;
3. If $n = 4$, there are many simply-connected closed manifolds that admit infinitely many distinct smooth structures; there are no smooth 4-manifolds known to have only finitely many smooth structures.

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Intersection forms

The study of manifold topology relies on intersection forms largely, which we now explain.

We can define the cup product pairing for closed oriented manifold M^4 as follows:

$$Q : H^2(M;\mathbb{Z}) \otimes H^2(M;\mathbb{Z}) \to \mathbb{Z}, (\alpha,\beta) \mapsto (\alpha \cup \beta)[M]$$

Due to Poincaré duality, the pairing is nonsingular when torsion is factored out. We focus on the case that M^4 is an oriented simply-connected closed manifold where $H^2(M;\mathbb{Z})$ is free.

Question: **To what extent can intersection form determine the topology and differential topology of manifolds?**

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Algebraic Classfication of unimodular forms

For indefinite unimodular forms, Hasse and Minkowski made a complete classfication. There are actually only **two** different cases, depending on whether it is odd or even.

But the classfication of definite forms is still open. Here we want to point out that there are **enormous** distinct definite forms, say at least 8 millions in rank 32, at least 10^{51} in rank 40.

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Topological Classfication

Freedman solved the classification problem completely.

Theorem (Freedman,1982)

For any integral symmetric unimodular form Q, there is a closed simply-connected topological 4-manifold that has Q as its intersection form. More precisely,

1. *If Q is even, there is exactly one such manifold;*
2. *If Q is odd, there are exactly two such manifolds, at least one of which doesn't admit any smooth structures.*

In particular, two smooth simply-connected 4-manifolds with isomorphic intersection forms are homeomorphic.

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Donaldson's Result

However, Donaldson proved the following surprising result:

Theorem (Donaldson,1983)

The only definite forms that can be realized by smooth 4-manifolds are just $\oplus m(1)$ and $\oplus m(-1)$.

Thus, none of exotic definite forms can be realized by smooth 4-manifolds.

Donaldson's Result

Several results can be deduced from Donaldson's theorem:

1. Any smooth simply-connected 4-manifold is homeomorphic to $\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ or $\#m\mathcal{M}_8\#n(S^2)$
2. A large number of topological 4-manifolds cannot admit a smooth structure
3. Existence of fake $\mathbb{R}^4$

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Sketch of Donaldson's proof

We sketch how Donaldson proved this remarkable result via instanton moduli space. He considered the $SU(2)$-bundle with Pontrjagin index -1 over M, and study the topology of instanton moduli space $\mathcal{M}$.

It turns out that $\mathcal{M}$ is a smooth 5-manifold with m singularities, and around these singularities are like a cone in $\mathbb{C}P^2$. Here m is half the solutions to $Q(\alpha, \alpha) = 1$. Moreover, $\mathcal{M}$ contains a collar of M such that $\overline{\mathcal{M}} = \mathcal{M} \cup M$ is a compact oriented smooth manifold with boundary. Therefore, **M is oriented cobordant to m disjoint complex projective spaces**, i.e.$\pm\mathbb{C}P^2 \amalg ... \amalg \pm\mathbb{C}P^2$.

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The final attack comes from a simple algebraic lemma:

Lemma

Let Q be a positive definite symmetric unimodular form with rank r, m is defined as before. Then $m \leq r$, where the equality holds if and only if Q is diagonalizable over $\mathbb{Z}$.

Sketch of Donaldson's proof

Now since the signature of intersection form is an oriented cobordism invariant, it follows that:

$$r = \sigma \leq m\sigma(\mathbb{C}P^2) = m$$

But from lemma, $m \leq r$. Hence the equality holds and Q is diagonalizable.**Q.E.D.**